ALGORITHM OF THE FINITE ELEMENTS METHOD OF SOLVING THREE-DIMENSIONAL HYDRODYNAMICS PROBLEMS

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An algorithm is proposed for computing velocity fields in channels on the basis of the joint application of finite-elements and finite-difference methods by using an exponential approximation of the desired functions in the elements.

In recent years the finite elements method (FEM) has become one of the most extensively utilized methods of numerical analysis for different physical processes. Its domain of application has been extended significantly from mechanics and heat conduction problems to complex heat transfer and hydrodynamics problems. New algorithms and methods for its realization have appeared, a modern mathematical support has been created to assure effective realization of the method on electronic computers [1, 2]. At the same time a traditional polynomial approximation of the desired functions in the elements is used in the majority of cases of FEM application to solve hydrodynamics problems together with the effective principles of constructing algorithms of the solution (penalty function method [3, 4], combined application of FEM and the method of characteristics [4, 5], etc.), which results in complexities analogous to the case of central-difference approximations of convective terms in transport equations by the finite-difference method (necessity of diminishing the mesh Reynolds numbers to obtain a stable solution, etc.).

An important step in the development of the method in application to two-dimensional problems of convective transport was the paper [6] where a new formulation was proposed for the method on the basis of the concept of a control volume as well as the approximation of functions being constructed in particular solutions of the convective diffusion part of the original equation. This permitted approximation of the properties of the finite-element scheme to the properties of exponential difference schemes.

However, as the practice of applying such an approach to the solution of hydrodynamics problems in domains with an abrupt change in geometry or with large source terms in the equations (problems with free convection taken into account for large Gr numbers, say) showed, it is possible to obtain non-physical solutions characterized by velocity and pressure fluctuations. The reason for this latter is source terms in the transport equations are not taken into account in the approximation [6] and a linear approximation for the pressure in the elements is used simultaneously.

An algorithm is proposed in this paper for the solution of three-dimensional problems of hydrodynamics by using FEM in the control volume formulation with the approximation of functions in the elements that takes account of the source terms of the motion equations. Let us elucidate the fundamental situations of this algorithm for the case of parabolized laminar viscous incompressible fluid flow in a channel of arbitrary section. The appropriate boundary value problem in dimensionless form is

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial X_j}=\frac{1}{\mathrm{Re}}\sum_{j=1}^{2}\frac{\partial^2 U_i}{\partial X_j^2}-\frac{\partial P}{\partial X_i}+F_i,\quad i=1,\ 2,\ 3; \tag{1}$$

$$\frac{\partial U_j}{\partial X_j}=0\quad \{t>0;\ X_1,\ X_2\in D;\ X_3>0\} \tag{2}$$

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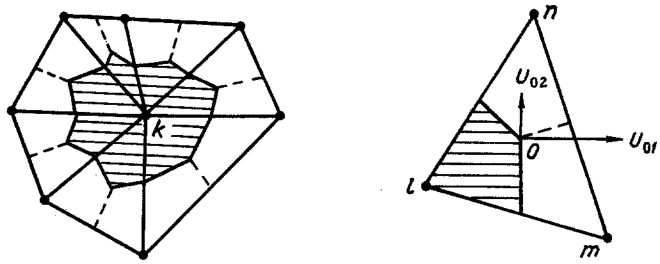


Fig. 1. Construction of control volumes on a triangular finite-element mesh.

with the single-valuedness conditions

$$\begin{cases} U_i|_{t=0} = U_{Hi}(X_1, X_2, X_3); \quad P|_{t=0} = P_H(X_1, X_2, X_3); \\ U_i|_{X_3=0} = U_{Bi}(X_1, X_2, t); \quad U_i|_{X_1, X_2 \in G} = 0. \end{cases} \tag{3}$$

Here P is represented in the form of the sum of functions $p(X_1, X_2, t)$, the pressure in the plane of the channel section, and $\bar{p}(X_3, t)$, the pressure averaged over the channel section [7].

To solve the problem (1)-(3), we use finite-element and finite-difference methods jointly, as in [8]. Since the original system of equations is nonlinear, then the algorithm for its solution will be iterative in nature. It is here constructed so that the continuity equation is satisfied for each iteration and the cofactors U_j ($j = 1, 2, 3$) in the convective terms are taken from the preceding iteration.

Executing discretization of the domain D by triangular elements and constructing control volumes on their basis (Fig. 1), we write the system (1) for an individual element by considering that U_j ($j = 1, 2, 3$) in the convective terms and F_i equal the mean values over the element (subscript "0")

$$\frac{\partial U_i}{\partial t} + U_{0j}\frac{\partial U_i}{\partial X_j} = \frac{1}{\mathrm{Re}}\sum_{j=1}^{2}\frac{\partial^2 U_i}{\partial X_j^2} - \frac{\partial P}{\partial X_i} + F_{0i}, \quad i = 1, 2, 3. \tag{4}$$

Here the subscripts for the number of the iteration (because of the remark made above) and the number of the element are omitted.

Let us consider the pressure P on an element to vary linearly, then we determine derivatives $\partial P/\partial X_i = P_{xi}$ = const also at the center of mass of the triangular element. To construct an approximation of the functions U_j in a finite element we consider the equation

$$\sum_{j=1}^{2}\left(U_{0j}\frac{\partial U_i}{\partial X_j} - \frac{1}{\mathrm{Re}}\frac{\partial^2 U_i}{\partial X_j^2}\right) = -P_{xi} + F_{0i}. \tag{5}$$

The functions

$$\xi_j = \frac{1}{\mathrm{Re}\,U_{0j}}\left(\exp\left[\mathrm{Re}\,U_{0j}(X_j - X_{0j})\right] - 1\right), \quad j = 1, 2;$$

$$\eta = \frac{1}{\mathrm{Re}\,W_0}\left(\prod_{j=1}^{2}\exp\left[\mathrm{Re}\,U_{0j}(X_j - X_{0j})/W_0\right] - 1\right),$$

$$\varphi = [U_{01}(X_2 - X_{02}) - U_{02}(X_1 - X_{01})]/W_0, \quad W_0 = \left(\sum_{j=1}^{2} U_{0j}^2\right)^{1/2};$$

will be particular solutions of the homogeneous equation (5), and also α = const. Let us represent the general solution of (5) in the form

$$U_i = Q_{0i}\psi + \alpha + \beta\xi_1 + \gamma\xi_2, \tag{6}$$

where α, β, γ are considered functions of the variables X_3, t; $Q_{0i} = -P_{xi} + F_{0i}$; ψ is a particular solution of (5) for $Q_{0i} = 1$.

The functions

$$Y = -\frac{1}{2}\mathrm{Re}\,\varphi^2; \quad Z = \frac{1}{W_0^2}\sum_{j=1}^{2} U_{0j}(X_j - X_{0j}),$$

say, can be taken as U_{0j} (j = 1, 2) however, it is expedient to use the former since as U_{0j} (j = 1, 2) tends to zero ($W_0 \to 0$) it tends to one of the particular solutions of the heat conduction equation with a single source, i.e., reflects the physics of the problem. The function Z will have no finite limit. Let us also note that in contrast to [6], a locally one-dimensional exponential approximation of the solution is used here. Use of the functions η and φ (see [6]) in the construction of (6) induces a character of linear behavior of the velocity vector components. Upon selection of ξ_j (j = 1, 2) as base results of the numerical experiments performed for the solution of different problems of hydrodynamics with the use of the approximation from [6] are taken into account also.

The coefficients α, β, γ in (6) are found for each element as in the case of the linear approximation of the solution, from the condition that U_i equal the nodal values of the desired quantities.

We now turn to examination of a sequence of calculations. Let us represent the functions U_i (i = 1, 2, 3), p, $\partial\bar{p}/\partial X_3$ in the form [8]

$$U_i = U_i' + U_i'' \quad (i = 1,\ 2,\ 3); \quad p = p' + p''; \qquad \frac{\partial\bar{p}}{\partial X_3} = \frac{\partial\bar{p}'}{\partial X_3} + \frac{\partial\bar{p}''}{\partial X_3}, \tag{7}$$

where p', $\partial\bar{p}'/\partial X_3$ are a given (from the preceding iteration, say) pressure distribution and its gradient to which the velocity distributions U_i' (i = 1, 2, 3) correspond. Using (6), integrating over the control volumes [6] and summing over all elements, we obtain a system of partial differential equations in the time and the coordinate X_3 with nonsymmetric tape matrices to determine the nodal values U_i' (i = 1, 2, 3) and which we solve by using an implicit difference scheme [8] by considering the velocity distribution at the preceding time and in the uspstream section already known. Having determined the nodal unknowns U_i' (i = 1, 2, 3), we correct the solution. We find the correction $\partial\bar{p}''/\partial X_3$ from constancy of the mass flow through the channel section at a given time, as in [8]. We find the desired pressure gradient from (7) and determine U_3 corresponding to $\partial\bar{p}/\partial X_3$, from the system (1).

To avoid the procedure of inverting the matrix and in order to save electronic computer storage, the following iteration scheme for finding the correction p" is proposed in contrast to the algorithm from [8]. We represent p" as the sum $p'' = p^* + \delta p$ and we write the equation for U_i'' (i = 1, 2)

$$\frac{U_i''}{\Delta t} + \sum_{j=1}^{2} U_j \frac{\partial U_i''}{\partial X_j} + U_3 \frac{U_i''}{\Delta X_3} = \frac{1}{\mathrm{Re}} \sum_{j=1}^{2} \frac{\partial^2 U_i''}{\partial X_j^2} - \frac{\partial p''}{\partial X_i}. \tag{8}$$

Let us represent U_i'' in an element in the form

$$U_i'' = \delta U_i'' + N_l U_{il}^* + N_m U_{im}^* + N_n U_{in}^* \quad (i = 1,\ 2), \tag{9}$$

where N_j (j = ℓ, m, n) are basis functions of the element obtained by using (6) for $p_{Xi} = \partial p^*/\partial X_i$; ℓ, m, n are local number of the nodes in the element, $\delta U_i''$ are particular solutions of an element analog of the system (8) of the form $\delta U_i'' = \frac{\partial(\delta p)}{\partial X_i} \Big/ \left(\frac{1}{\Delta t} + \frac{U_{03}}{\Delta X_3}\right)$; and U_i^* is a solution of (8) corresponding to the pressure p^*. To determine δp we substitute the approximation (9) into the continuity equation, we obtain the Poisson element equation for the pressure δp:

$$\sum_{j=1}^{2} \frac{\partial^2(\delta p)}{\partial X_j^2} = \left(\frac{1}{\Delta t} + \frac{U_{03}}{\Delta X_3}\right)\left[\sum_{j=1}^{2} \frac{\partial(U_j' + U_j^*)}{\partial X_j} + \frac{\partial U_3}{\partial X_3}\right]. \tag{10}$$

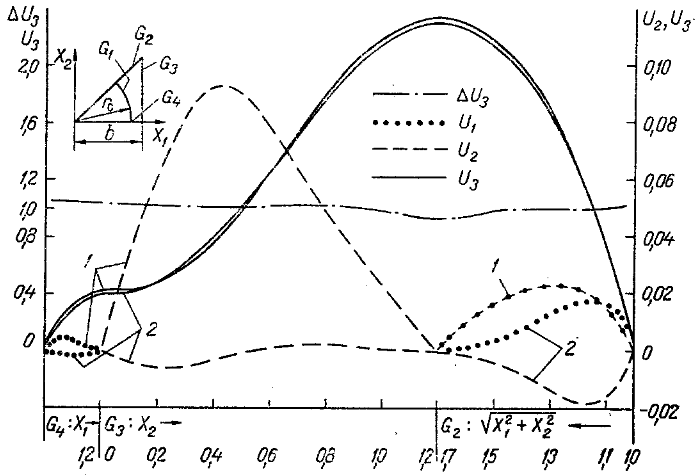
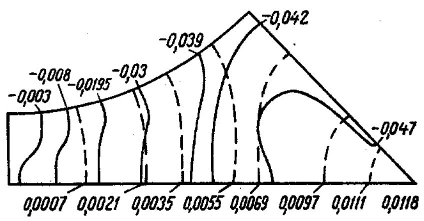


Fig. 2. Distribution of the velocity components U_i $(i = 1, 2)$ over the boundary of a translation element of a cascade of rods: 1) $X_3 = 0.1$, 2) $X_3 = 0.2$ (absolute values of U_1 and U_2 for $X_3 = 0.2$ enlarged ten-fold)



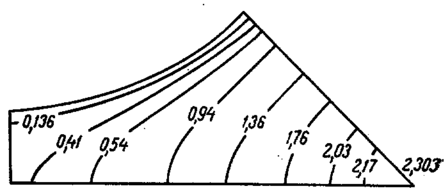


Fig. 3. Pressure isolines. Solid lines for the section $X_3 = 0.1$ and dashes for 0.2.

Fig. 4. Isolines of the velocity component U_3.

Summing (10) over all elements results in a system of algebraic equations with a symmetric tape matrix. Since the boundary condition for δp will be $\partial(\delta p)/\partial\bar{n}|_G = 0$, then it is necessary to give the pressure in a certain point of the domain D when solving this system.

Determining the nodal values of δp we find the new representation of p^* as $p^*_{k+1} = p^*_k + \delta p$ ($p^* = 0$ for $k = 0$, k is the number of the iteration in the pressure) and the computation is continued until the condition $\max\limits_{D}|\delta p| \leq \varepsilon_p$ is satisfied. It is clear that as $\delta p \to 0$ $p^* \to p''$, $U^* \to p''$, $U^* \to U''$, $\operatorname{div} \mathbf{U} \to 0$. Finding p'' and U''_i ($i = 1, 2$) in such manner, we determine the desired field U_i, p by means of (7).

The general iteration process for the channel computation section is continued until given accuracy of the results on the velocity field is achieved.

The algorithm described above with the locally one-dimensional exponential approximation of the velocity field and with the source terms of the transport equation taken into account is realized in the form of a set of programs of the solution on an electronic computer for nonstationary three-dimensional hydrodynamics problems in channels of complex section. It is approved by the solution of different test problems (fluid flow in a rectangular channel, say) and yielded better agreement (up to 3%) of the computed results with the known stationary solutions [7].

A stationary problem to determine the velocity field (Re = 50) in the beginning hydrodynamic section was solved in a cascade of rods with a square component for the cascade step $s = b/r_0 = 1, 2$. The velocity profile at the entrance to the channel was given homogeneous, and a translation element of the cascade was considered as the domain D (see Fig. 2). The boundary conditions for the velocity components were given in the form

$$U_i|_{X_3=0}=0,\quad U_3|_{X_3=0}=1\quad (i=1,\ 2;\ X_1,\ X_2\in D);$$
$$U_i|_{X_1,X_2\in G_1}=\left.\frac{\partial U_i}{\partial \bar{n}}\right|_{X_1,X_2\in G_2}=0\quad (i=1,\ 2,\ 3;\ X_3>0);$$
$$U_1|_{X_1,X_2\in G_3}=\left.\frac{\partial U_i}{\partial \bar{n}}\right|_{X_1,X_2\in G_3}=0\quad (i=2,\ 3;\ X_3>0);$$
$$U_2|_{X_1,X_2\in G_4}=\left.\frac{\partial U_i}{\partial \bar{n}}\right|_{X_1,X_2\in G_4}=0\quad (i=1,\ 3;\ X_3>0).$$

Distributions of the velocity vector components U_i ($i = 1, 2, 3$) over the sweep of the computed domain boundary $G_2 \cup G_3 \cup G_4$ are represented in Fig. 2. It is seen from the figure that an intensive efflux of fluid from the rod surface is observed in the section $X_3 = x_3/(r_0 Re) = 0.1$ in the domain of the maximum of the component U_3 caused by reforming of the velocity profile near the entrance section. The flow intensity in the plane D is reduced considerably in the second section. As computations showed, profile formation practically terminated in the section $X_3 = 0.4$ whose coordinate could be considered the length of the beginning hydrodynamic section in the cascade of rods under consideration ($L = 20r_0$).

Isolines of the field p in the section $X_3 = 0.1$, $X_3 = 0.2$ and of the velocity component U_3 in the section $X_3 = 0.5$ (stream stabilization section), constructed on an electronic computer from computed data, are represented in Figs. 3 and 4 to illustrate the flow pattern.

To confirm the reliability of the results obtained, the change in the quantity ΔU_3 equal to the ratio of the computed values of U_3 in the section $X_3 = 0.5$ to the values of $\bar{U}_3$ calculated by an approximate analytic solution of an analogous problem for the stabilized flow case [9] is shown in Fig. 2. It follows from the figure that the absolute values of ΔU_3 for practically the whole section G is almost 1. The maximal difference between U_3 and $\bar{U}_3$ (~8%) is observed in the region of their maximal values. However, it is necessary to note here that four terms of the series are taken into account in the analytic solution [9] and the values of $\bar{U}_3$ obtained on its basis in the domain of G_2 and G_3 intersection are somewhat exaggerated. Moreover, for $X_3 = 0.5$ the value of the grouping ξRe_e, obtained equal to 80.2, is computed and differs by 3.2% from that presented in [9].

In conclusion, let us note that the computations were performed on a mesh of 1372 elements containing 750 nodes.

NOTATION

$t = \tau w_0/r_0$, time; w_0 mean fluid flow velocity; $X_i = x_i/r_0$; r_0, characteristic dimension of the channel section domain D; $Re = w_0 r_0/\nu$; ν, kinematic viscosity; $P = P/(\rho w_0^2)$; ρ, density; F_i, mass force component in the X_i, axis direction; G, boundary of the domain D; $\bar{n}$, external normal to the boundary G; Δt, ΔX_3, steps in the difference mesh in the time and the coordinate X_3, respectively; εp, given accuracy of the pressure field calculations; ξ, friction drag coefficient, and $Re_e = w_0 d_e/\nu$; $d_e = 2r_0[4s^2/\pi - 1]$, equivalent diameter of the cascade.

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